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Modelling criteria for long water waves

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Model equations which describe the evolution of long-wave initial data in water of uniform depth are tested to determine explicit criteria for their applicability. We consider linear and nonlinear, dispersive and non-dispersive equations. Separate criteria emerge for the leading wave and trailing oscillations of the evolving wave train. The evolution of the leading wave depends on two parameters: the volume (non-dimensional) of the initial data and an Ursell number based on the amplitude and length of the initial data. The magnitudes of these two parameters determine the appropriate model equation and its time of validity. For the trailing oscillatory waves, a local Ursell number based on the amplitude of the initial data and the local wavelength determines the appropriate model equation. Finally, these modelling criteria are applied to the problem of tsunami propagation. Asymptotic ($t \to \infty$) linear dispersive theory does not appear to be applicable for describing the leading wave of tsunamis. If the length of the initial wave is approximately 100 miles, the leading wave is described by a linear non-dispersive model from the source region until shoaling occurs near the coastline. For smaller lengths (~ 40 miles) a linear dispersive (but not asymptotic) model is applicable. The longer-period oscillatory waves following the leading wave, which can induce harbour resonance, apparently require a nonlinear dispersive model.

1. Introduction

Despite the long history of water-wave investigations there is still little agreement regarding the appropriate model equations for describing the motion of long two-dimensional water waves of fairly small amplitude propagating in shallow water. Such problems often arise in the study of ocean waves such as tsunamis or storm surges and in the scaled models of these phenomena in the laboratory. Typically, the ratio η_0/h of the maximum wave amplitude to the depth is small, as is the reciprocal $(l/h)^{-1}$ of the dimensionless wavelength. Under these conditions, the initial propagation of two-dimensional surface waves is modelled by the linear wave equations

$$\eta_t + hu_x = 0, \quad u_t + g\eta_x = 0, \tag{1}$$

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where $\eta(x,t)$ is the wave height above the still water level and $u(x,t)$ is the horizontal fluid velocity. Equations (1) split an initial disturbance into left-running and right-running waves. Each of these two waves then develops independently (to leading order), even if the wave propagates beyond the time interval in which (1) is valid. The disagreement occurs in describing the evolution of these waves after this time.

The Korteweg–de Vries (1895) equation

$$(gh)^{-\frac{1}{2}}\eta_t + \eta_x + \tfrac{3}{2}h^{-1}\eta\eta_x + \tfrac{1}{6}h^2\eta_{xxx} = 0 \tag{2}$$

describes the evolution of right-running waves in a fluid of uniform depth. The problem with this model, of course, is its complexity. Simpler models are often obtained:

(i) by dropping the nonlinear term in (2), to yield a linear dispersive equation

$$(gh)^{-\frac{1}{2}}\eta_t + \eta_x + \tfrac{1}{6}h^2\eta_{xxx} = 0; \tag{3}$$

(ii) by dropping the dispersive term in (2) to yield a nonlinear non-dispersive equation

$$(gh)^{-\frac{1}{2}}\eta_t + \eta_x + \tfrac{3}{2}h^{-1}\eta\eta_x = 0, \tag{4}$$

(iii) by dropping both terms, to yield a linear non-dispersive equation

$$(gh)^{-\frac{1}{2}}\eta_t + \eta_x = 0, \tag{5}$$

which is equivalent to using (1). Models equivalent to these have been used by Carrier (1971) for (3), Airy (1845) for (4) and Hwang & Divoky (1970) for (5). Ursell (1953) showed that the parameter

$$U = \eta_0 l^2/h^3, \tag{6}$$

which we shall call the Ursell number, determines which of these models is appropriate; one must use (2), (3) or (4) depending on whether the Ursell number is of order unity, small or large, respectively. Ursell also suggested that a time-dependent Ursell number (based on the local characteristics near the front of the evolving wave train) will tend towards an order-one limit, so that all waves in this category eventually propagate according to (2). (Ursell actually used slightly different models which are asymptotically equivalent to those used herein.)

There seems to be no disagreement with Ursell's results; however, there is disagreement about how to measure the length which appears in (6), how to interpret 'order unity' and how definitive Ursell's results actually are. The purpose of this paper is to extend Ursell's argument to obtain definitive criteria which determine how to model a given problem. As a specific application of these criteria, the problem of tsunami propagation will be examined.

In order to obtain definite results the following point of view is adopted. If the initial disturbance is sufficiently smooth and localized, and if η_0/h and h/l, based on the initial data, are both small, then (2) will be the appropriate model eventually. Equations (3), (4) and (5) can be considered as approximate models, whose solutions approximate the solution of the KdV equation (2) for a limited time. By comparing these times with the duration of the problem in question (e.g. the time of propagation of a tsunami between a source and target region, or of a wave down the length of a laboratory tank) one can determine whether the given model is adequate for the problem in question.

As shown below, this analysis provides the criteria, based on the overall dimensions of the initial disturbance, necessary to determine which theory models the leading wave that evolves from the initial data. For engineering applications, this wave is often the one of interest. However, the initial disturbance can also generate a train of dispersive oscillatory waves which follow this leading wave. In this region only (2) and (3), which are dispersive, can provide any information regarding wave evolution. By comparing the asymptotic ($t \to \infty$) solutions of these equations, a different criterion, based on a local Ursell number, is found to determine whether the solution of (3) is an adequate approximation of the solution of the KdV equation.

Two points are worth noting. First, the criteria obtained here are based on a comparison of the solutions of (2), (3), (4) and (5), which makes these results somewhat more definitive than those of Ursell, who simply compared the equations. Second, much of the analysis reported herein has been performed elsewhere. The contribution of this paper is to fit these pieces together, in order to obtain explicit results.

2. Analysis of leading wave

2.1. *Linear dispersive theory*

In order to compare (2) and (3), it is convenient to rewrite them by defining new variables

$$\chi = h^{-1}[x-(gh)^{\frac{1}{2}}t], \quad \tau = \tfrac{1}{6}(g/h)^{\frac{1}{2}}t, \quad f(\chi,\tau) = \tfrac{3}{2}h^{-1}\eta(x;t). \tag{7}$$

Using these variables, the KdV equation becomes

$$f_\tau + 6ff_\chi + f_{\chi\chi\chi} = 0 \tag{8}$$

and (3) becomes

$$f_\tau + f_{\chi\chi\chi} = 0. \tag{9}$$

In either case, the initial data are given by

$$f(\chi,0) \equiv \phi(\chi) \equiv \tfrac{3}{2}h^{-1}\eta(x,0), \tag{10}$$

where $\phi(\chi)$ is a smooth function that vanishes rapidly as $|\chi| \to \infty$.

In terms of these variables, the linear dispersive theory is valid as long as the solution of (9) approximates that of (8). In practical terms, however, the major advantage of the linear dispersive theory is that its solution achieves a simple asymptotic form as $\tau \to \infty$. For short times, (9) offers no particular advantage over (8); hence the primary question for linear dispersive theory is: during what time interval does the asymptotic ($\tau \to \infty$) solution of (9) approximate the solution of (8)? The answer to this question is given by inequality (28) below, which is the consequence of the analysis up to that point.

To determine this time interval, parameters which characterize the initial data will be required. Let

$$\bar{V} = \int_{-\infty}^{\infty} \phi(\chi)\,d\chi \tag{11}$$

be the dimensionless volume of the disturbance. If L denotes the characteristic length of the initial data, then

$$U_0 = (L/h)|\bar{V}| \tag{12}$$

is an Ursell number based on the overall dimensions of the initial data. (It is assumed that $\bar{V} \neq 0$; the results of the analysis change significantly when $\bar{V} = 0$.) The maximum initial wave amplitude is η_0. Since none of the model equations used herein are valid for breaking waves, we require

$$\eta_0/h < 1. \tag{13}$$

Another characteristic wave amplitude a can be defined by

$$a/h = (h/L)\,\bar{V}. \tag{14}$$

Necessarily,

$$a/h \leqslant \tfrac{3}{2}\eta_0/h$$

but if the longest waves comprise most of the initial data, as in the case considered herein, these two amplitudes are approximately equal.

Assume that there is a small parameter ϵ associated with the amplitude of the initial data. (The required definition of ϵ will come out of the analysis.) Consequently, a formal series solution of (8) is sought in the form

$$f = \epsilon f_1 + \epsilon^2 f_2 + O(\epsilon^3). \tag{15}$$

Substituting (15) into (8) yields a hierarchy of problems:

$$(f_1)_\tau + (f_1)_{\chi\chi\chi} = 0, \quad \epsilon f_1(\chi, 0) = \phi(\chi); \tag{16}$$

$$(f_2)_\tau + (f_2)_{\chi\chi\chi} = -6 f_1 (f_1)_\chi, \quad f_2(\chi, 0) = 0; \tag{17}$$

etc.

The solution of (16) which is equivalent to (9) is well known:

$$\epsilon f_1(\chi, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(k) \exp[i(k\chi + k^3\tau)]\,dk, \tag{18}$$

where

$$\hat{\phi}(k) = \int_{-\infty}^{\infty} \phi(\chi) \exp(-ik\chi)\,d\chi$$

is the Fourier transform of the initial data. As $\tau \to \infty$, $|\chi|/\tau \to 0$, the asymptotic form of this solution is

$$\epsilon f_1(\chi, \tau) = \hat{\phi}(0)(3\tau)^{-\frac{1}{3}}\mathrm{Ai}(\zeta) - i\hat{\phi}'(0)(3\tau)^{-\frac{2}{3}}\mathrm{Ai}'(\zeta) - \tfrac{1}{2}\hat{\phi}''(0)(3\tau)^{-1}\mathrm{Ai}''(\zeta) + O[(3\tau)^{-\frac{4}{3}}], \tag{19}$$

where

$$\zeta = \chi/(3\tau)^{\frac{1}{3}} \tag{20}$$

and $\mathrm{Ai}(\zeta)$ is the Airy function defined by

$$\mathrm{Ai}(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i(\kappa\zeta + \tfrac{1}{3}\kappa^3)]\,d\kappa.$$

(The region defined by $-\chi/\tau = O(1)$ is also of interest and will be treated separately.) The coefficients in (19) have simple interpretations:

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi\,d\chi = \bar{V},$$

$$-i\hat{\phi}'(0) = \int_{-\infty}^{\infty} \chi\phi\,d\chi = C_1 U_0,$$

$$-\tfrac{1}{2}\hat{\phi}''(0) = \int_{-\infty}^{\infty} \chi^2\phi\,d\chi = C_2 \frac{U_0^2}{\bar{V}},$$

where C_1 and C_2 are constants that depend on details of the initial data. Thus (19) becomes (as $\tau \to \infty$)

$$\epsilon f_1(\chi, \tau) = \bar{V}(3\tau)^{-\frac{1}{3}}\left\{\mathrm{Ai}(\zeta) + C_1\left(\frac{U_0}{\bar{V}}\right)(3\tau)^{-\frac{1}{3}}\mathrm{Ai}'(\zeta) + C_2\left(\frac{U_0}{\bar{V}}\right)^2(3\tau)^{-\frac{2}{3}}\mathrm{Ai}''(\zeta) + O\left[\left(\frac{U_0}{\bar{V}}\right)^3(3\tau)^{-1}\right]\right\}. \quad (21)$$

From (21) it follows that the time required for this representation to become asymptotic is at least

$$(3\tau)^{\frac{1}{3}} \gg U_0/|\bar{V}|. \quad (22)$$

A particular solution of (17) is

$$\epsilon^2 f_{2p}(\chi, \tau) = -\left[\int_{-\infty}^{\chi} \epsilon f_1(z, \tau)\, dz\right]^2. \quad (23)$$

Defining

$$\Theta(\chi) = \frac{1}{\bar{V}^2}\left[\int_{-\infty}^{\chi} \phi(z)\, dz\right]^2,$$

the homogeneous solution of (17) is

$$\epsilon^2 f_{2h}(\chi, \tau) = \bar{V}^2\int_{-\infty}^{\zeta} \mathrm{Ai}(z)\, dz + \frac{\bar{V}^2}{2\pi}\int_{-\infty}^{\infty}\{\hat{\Theta}(k) - (ik)^{-1}\}\exp[i(k\chi + k^3\tau)]\, dk. \quad (24)$$

Thus as $\tau \to \infty$, $|\chi|/\tau \to 0$

$$\epsilon^2 f_2(\chi, \tau) = \bar{V}^2\left\{-\left[\int_{-\infty}^{\zeta} \mathrm{Ai}(z)\, dz\right]^2 + \int_{-\infty}^{\zeta} \mathrm{Ai}(z)\, dz + O[(3\tau)^{-\frac{1}{3}}]\right\}. \quad (25)$$

The appropriate definition of ϵ is found from (21) and (25):

$$\epsilon = \bar{V} \equiv \frac{3}{2h^2}\int_{-\infty}^{\infty} \eta(x, 0)\, dx. \quad (26)$$

This result has two important consequences. First, linear dispersive theory (along with its large-time asymptotics) is ordinarily justified on the basis of small wave amplitudes. It is seen from (26) that the correct justification must be that the dimensionless *volume* is small, and arguments based on wave amplitude alone can be misleading. If the expansion is based on amplitude alone, the resulting solution does not satisfy the initial and boundary conditions of (17). Second, laboratory experiments on long waves of small amplitude should be scaled to preserve the dimensionless wave volume or these results may also be misleading.

Note that the solution (25) remains $O(1)$ as $\tau \to \infty$, so that the series (15) cannot be asymptotic after

$$(3\tau)^{\frac{1}{3}} = O(|\bar{V}|^{-1}). \quad (27)$$

Thus, if an initial wave has a dimensionless volume $\bar{V}$ ($\neq 0$) and an Ursell number U_0 based on its overall dimensions, and if both of these parameters are small, then asymptotic linear dispersive theory is valid in an interval no larger than

$$U_0/|\bar{V}| \ll (3\tau)^{\frac{1}{3}} \ll 1/|\bar{V}|. \quad (28)$$

(It has been shown that the theory fails outside this interval; however, this does not necessarily mean that the theory is valid within the entire interval.) If either U_0 or $\bar{V}$ is not small, then asymptotic ($\tau \to \infty$) linear dispersive theory breaks down immediately.

2.2. *Non-dispersive theories*

The KdV equation is usually derived as a model of long water waves by assuming that there exist two time scales (see, for example, Benney 1966; Korteweg & de Vries 1895). Korteweg & de Vries essentially assumed

$$U = O(1) \tag{29a}$$

and found (1) or (5) on the short time scale. Equation (2) or (8) occurs over a long time scale defined by $T = (|a|/h)\,t$, where a/h is the dimensionless wave amplitude defined in (14). It follows that the maximum interval of validity of (1) or (5) is

$$3\tau \ll h/|a| = U_0/\bar{V}^2. \tag{29b}$$

The same result applies if $U_0 \gg 1$. If $U_0 \ll 1$, then $|\bar{V}| \ll 1$ as well, and linear dispersive theory is relevant. The maximum interval of validity of (1) or (5) for this case is

$$3\tau \ll (L/h)^2 = (U_0/\bar{V})^2. \tag{29c}$$

If $U_0 \gg 1$, then (4) provides the first correction after (1) has lost its validity. It is well known from the theory of characteristics that solutions of (4) become discontinuous after a finite time. Certainly these solutions no longer approximate the solution of the KdV equation after discontinuities form. The time required for these discontinuities to occur depends on the detailed structure of the initial data. The longest time possible is

$$3\tau \ll L/|a| = U_0^2/|\bar{V}|^3, \tag{30}$$

however the theory could become invalid well before this time.

Between them, (28), (29*b*, *c*) and (30) define the maximum ranges of validity of (3)–(5) as an approximation of the KdV equation. Beyond those times one must use the KdV equation itself (or something equivalent). It remains to be determined when the solution of this equation achieves its asymptotic form.

2.3. *Asymptotic KdV theory*

The 'inverse scattering transform' was discovered by Gardner *et al.* (1967, 1974) as a method for solving the KdV equation for initial data which are smooth and vanish rapidly as $|\chi| \to \infty$. The method uses the initial data $f(\chi, 0)$ for (8) as the potential in the linear eigenvalue problem

$$d^2\psi/d\chi^2 + [\lambda + f(\chi, 0)]\,\psi = 0. \tag{31}$$

The spectrum of eigenvalues can contain both discrete ($\lambda < 0$) and continuous ($\lambda \geq 0$) sets and each part of the spectrum contributes to the asymptotic ($\tau \to \infty$) solution of (8).

If $\bar{V} > 0$, the number N of discrete eigenvalues is approximately given by

$$N - 1 = O(U_0). \tag{32}$$

A more precise statement is provided in Segur (1973). Each of these eigenvalues ($\lambda_n = -\kappa_n^2$) contributes a soliton (a positive permanent wave)

$$f_n(\chi, \tau) = 2\kappa_n^2 \operatorname{sech}^2 \kappa_n (\chi - \chi_n - 4\kappa_n^2 \tau) \tag{33}$$

to the asymptotic KdV solution. The largest of these solitons travels the fastest, and eventually appears at the front of the wave. The time required for this wave to emerge is approximately its 'sorting time'

$$\tau_s = O(L/\eta_0) = O(U_0^2/\bar{V}^3), \tag{34}$$

as demonstrated by Hammack & Segur (1974). This is probably the earliest time that the solution can be described as an ordered set of solitons followed by a train of dispersive oscillatory waves. The amplitudes of these waves reach their final (constant) values at a somewhat later time.

The asymptotic contribution from the continuous spectrum ($\lambda = k^2$) is qualitatively similar to the solution of (9), as shown by Ablowitz & Segur (1976). (Their results apply only to cases in which there are no solitons. Because the asymptotic contributions from the discrete and continuous spectra dominate in different regions of space, it appears that their results could be applied to the general case with only minor modification. However, in the analysis which follows, it will be assumed that no solitons evolve.) The boundary conditions for (31) are

$$\psi(\chi) \sim \begin{cases} e^{-ik\chi} + r(k)\, e^{ik\chi}, & \chi \to +\infty, \\ [a(k)]^{-1} e^{-ik\chi}, & \chi \to -\infty, \end{cases} \tag{35}$$

where $r(k)$ and $1/a(k)$ are reflexion and transmission coefficients, respectively, for the potential $f(\chi, 0)$. The reflexion coefficient $r(k)$ plays the role of the 'nonlinear Fourier transform', as shown in Ablowitz *et al.* (1974).

For almost any initial data,

$$r(0) = -1, \tag{36}$$

which implies that the asymptotic solution of the KdV equation has the following behaviour as $\tau \to \infty$ (see Ablowitz & Segur 1976).

(i) $\chi/\tau = O(1)$. Here the solution is exponentially small. (If solitons were present, they would dominate in this region.)

(ii) $\zeta = \chi/(3\tau)^{\frac{1}{3}} = O(1)$. For $\zeta > 0$, the solution matches the exponentially small solution ahead of it. For $\zeta < 0$, it approaches the self-similar solution

$$f(\chi, \tau) \sim (3\tau)^{-\frac{2}{3}} \{\tfrac{1}{2}\zeta - \tfrac{1}{2}(-2\zeta)^{-\frac{1}{2}} + O[(-\zeta)^{-2}]\}. \tag{37}$$

For $\zeta \to -\infty$, the representation (37) breaks down, as smaller terms in the expansion grow exponentially. The solution here is

$$f(\chi, \tau) \sim (3\tau)^{-\frac{2}{3}} (-2\zeta) \left\{ -\tfrac{1}{4} - \tfrac{1}{2}(-2\zeta)^{-\frac{3}{2}} + \ldots + (3\tau)^{-\frac{1}{3}} \frac{KM}{(-2\zeta)^{\frac{5}{4}}} \exp[\tfrac{1}{3}(-2\zeta)^{\frac{3}{2}}] + \ldots \right\}, \tag{38}$$

where

$$M = \tfrac{1}{8}\{r''(0) - [r'(0)]^2\}, \tag{39a}$$

and K is a universal constant; approximately,

$$K = 0{\cdot}8. \tag{39b}$$

(iii) The representation (38) matches with a 'dissipationless shock layer', which then matches with a region of slowly varying oscillations, which are discussed below.

The maximum amplitude of the leading wave is found from (38):

$$f_{\max} \sim (3\tau)^{-\frac{2}{3}} (\tfrac{1}{2}\bar{\zeta}), \tag{40}$$

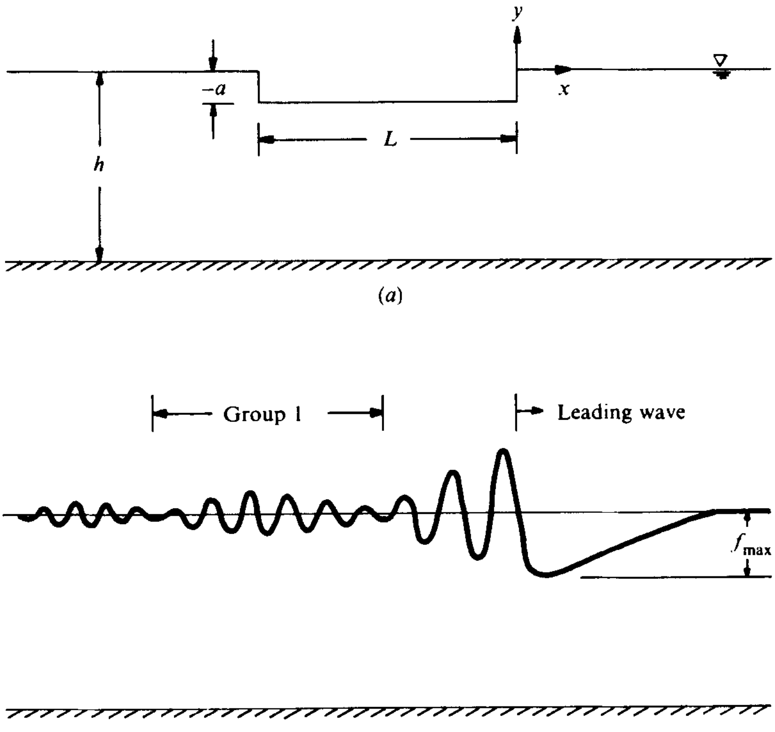

FIGURE 1. (a) Rectangular initial ($\tau = 0$) wave. (b) Asymptotic ($\tau \to \infty$) structure of initial waves from which no solitons evolve.

where $\bar{\zeta}$, the location of the extremum, is given by

$$(-2\bar{\zeta})^{\frac{3}{2}} = 2\ln 3\tau - \tfrac{3}{4}\ln(-2\bar{\zeta}) - 3\ln(2KM). \tag{41}$$

Figure 1 (b) shows the qualitative behaviour of a wave that has evolved from negative initial data (as in figure 1a) according to the KdV equation (8). Linear dispersive theory, (9), yields similar qualitative behaviour. (For a comparison of both of these theories with experimental data, see the companion paper Hammack & Segur 1978.)

The time required for this leading wave region to achieve its asymptotic state is, from (41) (by assuming $-\bar{\zeta}$ large),

$$(3\tau)^2 \gg (2KM)^3, \tag{42}$$

where M depends on the initial data. In order to relate (42) to U_0 and $\overline{V}$, attention is restricted to the particular class of initial data shown in figure 1(a): rectangular waves. If the amplitude of the rectangular wave is positive, most of the wave energy goes into solitons, and (34) is relevant. If the amplitude is negative, no solitons exist and (42) applies. For this case the reflexion coefficient is (see Schiff 1968, p. 103)

$$r(k) = \frac{\alpha \sin\{(k^2+\alpha)^{\frac{1}{2}}\Lambda\}}{(2k^2+\alpha)\sin\{(k^2+\alpha)^{\frac{1}{2}}\Lambda\} + 2ik(k^2+\alpha)^{\frac{1}{2}}\cos\{(k^2+\alpha)^{\frac{1}{2}}\Lambda\}}, \tag{43}$$

where $\alpha = a/h$, and $\Lambda = L/h$. For $\alpha < 0$, it follows from (39) that

$$M = [2|\alpha| \sinh^2(|\alpha|^{\frac{1}{2}}\Lambda)]^{-1}. \tag{44}$$

Note that

$$(|\alpha|^{\frac{1}{2}}\Lambda)^2 = |\alpha| L^2/h^3 = U_0. \tag{45}$$

Combining (42), (44) and (45) gives the time required for the leading (negative) wave in the KdV solution to obtain its asymptotic form. This time 3τ is as follows for various cases:

(i) $3\tau \leqslant O(1)$ for $U_0 \gg 1$, $U_0^{\frac{1}{2}} > \ln|U_0^{\frac{3}{4}}/\bar{V}|$; (46*a*)

(ii) $3\tau = O[(U_0^{\frac{3}{2}}/\bar{V}^3)\exp(-3U_0^{\frac{1}{2}})]$ for $U_0 \gg 1$, $U_0^{\frac{1}{2}} < \ln|U_0^{\frac{3}{4}}/\bar{V}|$; (46*b*)

(iii) $3\tau = O(|\bar{V}|^{-3})$ for $U_0 = O(1)$; (46*c*)

(iv) $3\tau = O(|\bar{V}|^{-3})$ for $U_0 \ll 1$. (46*d*)

Notice that a consequence of (46*a*, *b*) is that, for this class of initial data, the nonlinear non-dispersive theory (4) is never required.

2.4. *Summary of results for the leading wave*

Since several theories for the leading wave region have been considered, a summary of results for this region is appropriate. To obtain definite statements regarding the asymptotic KdV theory, attention has been restricted to rectangular initial data. Under this restriction, negative waves produce no solitons, while positive waves produce mostly solitons.

For negative waves ($\alpha < 0$) of small amplitude which are also long, five possible cases exist. Representative cases are shown in figure 2, where the asymptotic solution for the maximum amplitude f_{max} of the leading wave is shown as a function of 3τ.

(i) $U_0 \gg 1$, $U_0^{\frac{1}{2}} > \ln|U_0^{\frac{3}{4}}/\bar{V}|$. The KdV solution becomes asymptotic almost immediately and there is no reason to adopt any other theory (see figure 2).

(ii) $U_0 \gg 1$, $|\bar{V}| \ll 1$, $U_0^{\frac{1}{2}} < \ln|U_0^{\frac{3}{4}}/\bar{V}|$. The maximum interval of validity for the linear non-dispersive theory (5) is given by (29*b*), while the asymptotic KdV theory takes over according to (46*b*). If information is needed at intermediate times, the safest procedure is to compute the solution of the KdV equation numerically. An alternative procedure is to compute using nonlinear non-dispersive theory (4) until it breaks down, and then use the KdV equation.

(iii) $U_0 = O(1)$, $\bar{V} = O(1)$. The KdV solution becomes asymptotic almost immediately (see figure 2).

(iv) $U_0 = O(1)$, $|\bar{V}| \ll 1$. Linear non-dispersive theory (5) breaks down according to (29*b*), while the asymptotic KdV theory takes over according to (46*c*). For information at intermediate times, a numerical solution of the KdV equation is required.

(v) $U_0 \ll 1$, $|\bar{V}| \ll 1$. This is the only case for which linear dispersive theory (3) applies. Initially, propagation is modelled by linear non-dispersive theory, which breaks down according to (29*c*). Linear dispersive theory follows, its asymptotics becoming applicable according to (28). Asymptotic KdV theory takes over according to (46*d*). Note that the asymptotic KdV and linear dispersive theories can be patched together at $3\tau = O(|\bar{V}|^{-3})$ as shown in figure 2.

For positive waves ($\alpha > 0$), the results are slightly simpler.

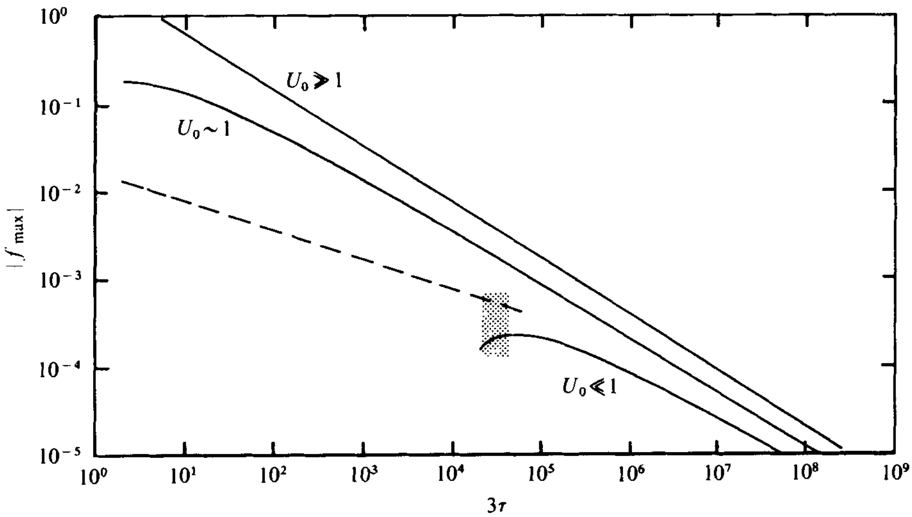

FIGURE 2. Asymptotic ($\tau \rightarrow \infty$) decay of leading wave for rectangular and negative initial data. ——, KdV asymptotics; – – –, linear dispersive asymptotics. Shaded area represents region where $3\tau = O(|\bar{V}|^{-3})$ and asymptotic solutions may be patched.

(i) $U_0 \gg 1$. There are many solitons, for which the sorting time is given by (34). For shorter times, the KdV equation must be integrated numerically. For these rectangular waves nonlinear non-dispersive theory breaks down immediately.

(ii) $U_0 = O(1)$. Linear non-dispersive theory (5) breaks down according to (29*b*), after which time only the KdV equation applies. The soliton description applies after the sorting time given by (34).

(iii) $U_0 \ll 1$. There is one soliton, followed by oscillatory waves. As in (v) above, asymptotic linear dispersive theory applies in an interval no larger than that given by (28), while the soliton can be identified after its sorting time, given by (34). Note that these two intervals overlap; hence the two theories may actually be matched for this case.

3. Analysis of trailing wave oscillations

Whereas most of the destructive energy of an evolving wave train may reside in its leading wave, the oscillatory waves which follow can also create severe problems if these waves happen to have frequencies equivalent to the natural frequency of a harbour ('harbour resonance') or other coastal configurations. Consequently, it is also of interest to examine the trailing oscillatory waves. Regardless of U_0, the Ursell number based on the overall dimensions of the initial data, these trailing waves are *ipso facto* dispersive. Hence the only relevant question for these waves is whether linear dispersive theory (3) can be used to approximate the KdV equation.

The solution of (9), linear dispersive theory, is still given by (18), which can be evaluated as $\tau \rightarrow \infty$ with $-\chi/\tau = O(1)$ by the method of stationary phase:

$$f(\chi,\tau) \sim (3\tau)^{-\frac{1}{2}}(4\pi|k|)^{-\frac{1}{2}}\{\hat{\phi}(k)\exp[i(\tfrac{2}{3}k^3(3\tau)+\tfrac{1}{4}\pi)]+\hat{\phi}(-k)\exp[-i(\tfrac{2}{3}k^3(3\tau)+\tfrac{1}{4}\pi)]\}, \tag{47a}$$

where

$$k^2 = -\chi/3\tau = O(1). \tag{47b}$$

In the same region, the asymptotic solution of the KdV equation is (Ablowitz & Segur 1976)

$$f(\chi,\tau) \sim (3\tau)^{-\frac{1}{3}}|k|^{\frac{1}{2}}\, 2d\cos\Theta, \tag{48a}$$

where

$$\Theta = \tfrac{2}{3}k^3(3\tau) - 2d^2\ln 3\tau + O(1), \tag{48b}$$

$$d^2 = -(4\pi)^{-1}\ln\{1-|r(\tfrac{1}{2}k)|^2\} \tag{48c}$$

and (47*b*) has been used. For $|r(\frac{1}{2}k)| \ll 1$, (48*c*) is approximated by

$$d^2 \simeq (4\pi)^{-1}\,|r(\tfrac{1}{2}k)|^2, \tag{49}$$

so that (48*a*) becomes

$$f(\chi,\tau) \sim (3\tau)^{-\frac{1}{3}}\,(4\pi|k|^{-\frac{1}{2}}\{2|k|r(\tfrac{1}{2}k)\cos\Theta\}. \tag{50}$$

Now the question becomes: under what circumstances will (47*a*) and (50) yield approximately the same results?

To obtain definite results, a negative rectangular wave as shown in figure 1(*a*) is again considered. Hence with $\alpha < 0$,

$$\phi(k) = \frac{\alpha\exp(-\frac{1}{2}ik\Lambda)\sin(\frac{1}{2}k\Lambda)}{\frac{1}{2}k} \tag{51}$$

while

$$kr(\tfrac{1}{2}k) = \frac{\alpha k\sin[(\frac{1}{4}k^2+\alpha)^{\frac{1}{2}}\Lambda]}{(\frac{1}{2}k^2+\alpha)\sin[(\frac{1}{4}k^2+\alpha)^{\frac{1}{2}}\Lambda]+ik(\frac{1}{4}k^2+\alpha)^{\frac{1}{2}}\cos[(\frac{1}{4}k^2+\alpha)^{\frac{1}{2}}\Lambda]}. \tag{52}$$

It can be verified that if

$$\tfrac{1}{4}k^2 \gg |\alpha| = |a|/h \tag{53}$$

then

$$kr(\tfrac{1}{2}k) \sim i\phi(k), \tag{54}$$

and the two results coincide to leading order. However, (53) can be written as

$$\tilde{U} \ll \tfrac{1}{4}, \tag{55a}$$

where

$$\tilde{U} = |\alpha|/k^2 \propto |a|l^2/h^3 \tag{55b}$$

is a local Ursell number based on the original wave amplitude and the local wavenumber. Thus, for the trailing wave oscillations, the linear theory approximates the nonlinear theory whenever the local Ursell number is small. In regions where (53) fails, the results of the two theories differ considerably. For the case of rectangular initial data, the dominant wavenumbers in the trailing wave groups are

$$k_m(\text{linear}) = (2m-1)\pi h/L, \quad m = 2,3,\ldots, \tag{56a}$$

$$k_m(\text{KdV}) = \{4|a|/h+[(2m-1)\pi h/L]^2\}^{\frac{1}{2}}, \quad m = 1,2,\ldots. \tag{56b}$$

These dominant wavenumbers are represented by local extrema in the amplitude spectra given by (51) and (52); hence the energy content of the oscillatory waves is concentrated in the vicinity of these k. A separate (node-to-node) wave group exists for each k_m, with $m = 1$ corresponding to the first group behind the leading wave for the nonlinear solution and $m = 2$ representing the first group in the linear solution. (The discrepancy in the numbering of wave groups for small m occurs because the linear and nonlinear solutions are required to yield the same results as $k\to\infty$ or $m\to\infty$.) A new feature indicated by (56*a*, *b*) is that the linear solution fails to predict the existence of an energy peak (or wave group) corresponding to $m = 1$ in the nonlinear solution. (See the companion paper Hammack & Segur 1978.)

4. Two-dimensional tsunamis: an example

As a specific application of these results, the problem of a two-dimensional tsunami propagating in an ocean of uniform depth is considered. A simplified model is used in order to select a basic theory, which can then be modified to include other effects such as variations in ocean depth and three-dimensional spreading of wave energy.

Using information from the Alaskan earthquake of 1964 as a guide (cf. Plafker 1969; Van Dorn 1964), typical initial dimensions for a destructive tsunami appear to be

$$a = 1\,\text{ft}, \quad h = 10^4\,\text{ft}, \quad L = 10^6\,\text{ft}. \tag{57a}$$

On the basis of these values, the controlling parameters for the leading wave are

$$U_0 = 1, \quad \overline{V} = 10^{-2}. \tag{57b}$$

Thus propagation of the leading wave in the open ocean is modelled by the linear wave equation (1) or (5) followed by the KdV equation (2). Equations (3) and (4) are inapplicable for the leading wave region. The breakdown of (1), as determined from (29*b*), occurs when

$$3\tau \simeq 10^4.$$

From (7), the distance X the leading wave has propagated during this time is given by

$$\tfrac{1}{2}X/h = 3\tau \simeq 10^4,$$

or

$$X \simeq 40000\,\text{miles}. \tag{58}$$

Consequently, neither nonlinearity nor frequency dispersion has any effect on the leading wave as it propagates across any ocean.

Alternatively, Carrier (1971) has used

$$a = 10\,\text{ft}, \quad h = 1{\cdot}5\times 10^4\,\text{ft}, \quad L = 2\times 10^5\,\text{ft}, \tag{59a}$$

which models a shorter tsunami of larger amplitude in deeper water compared with that of (57*a*). Using these values, one obtains

$$U_0 \simeq 10^{-1}, \quad \overline{V} \simeq 10^{-2} \tag{59b}$$

and the breakdown of (1) occurs, according to (29*c*), for

$$X \ll 600\,\text{miles}. \tag{60}$$

Hence linear frequency dispersion may affect this wave over much of typical ocean trajectories as evidenced by Carrier's (1971) computations. Even so, on the basis of (28), linear asymptotics apply only for

$$X \gg 6000\,\text{miles}, \tag{61}$$

which exceeds the length of relevant trajectories. Assuming the tsunami characteristics of (57*a*) and (59*a*) to span the range of realistic values, linear asymptotics are not applicable to describing the propagation of the leading wave ($k = 0$). Whether anything more complicated than (1) is required depends primarily on the length of the initial disturbance.

When the leading wave reaches a continental shelf, the depth is reduced significantly. In this region, if the tsunami characteristics based on (57*a*) are assumed with the depth reduced to $h = 500\,\text{ft}$, then

$$U_0 \simeq 10^4, \quad \bar{V} \simeq 4. \tag{62}$$

From (29*b*) it is seen that linear non-dispersive theory can be used until

$$3\tau \simeq 10^3,$$

and the distance across the continental shelf (of uniform depth) that the leading wave propagates during this time is given by

$$\tfrac{1}{2}X/h \simeq 10^3,$$

or

$$X = 200\,\text{miles}. \tag{63}$$

But most continental shelves are less than two hundred miles wide; hence for long ($L \sim 100$ miles) tsunamis linear non-dispersive theory is the relevant model for the leading wave from the source region to where the wave begins shoaling near a beach. For shorter tsunamis ($L \sim 40$ miles), the leading wave may be dispersive before striking the continental shelf. In either case, the appropriate theory over the shelf for specific cases can be found by the procedures outlined above.

In order to examine the trailing oscillatory waves of a tsunami, the wave characteristics of (57*a*) are assumed to represent a negative rectangular wave. Then, according to KdV asymptotics (56*b*), the first wave group ($m = 1$) trailing behind the leading negative wave has the dominant wavenumber $k_1 = 0{\cdot}037$. For subsequent wave groups the wavenumbers are found from (56*a, b*) to be

$$\left.\begin{aligned} k_2(\text{linear}) &= 0{\cdot}094, \quad k_2(\text{KdV}) = 0{\cdot}096,\\ k_3(\text{linear}) &= 0{\cdot}157, \quad k_3(\text{KdV}) = 0{\cdot}158,\\ k_4(\text{linear}) &= 0{\cdot}220, \quad k_3(\text{KdV}) = 0{\cdot}221.\end{aligned}\right\} \tag{64}$$

Thus the critical wavenumber k_c which represents the lower limit of applicability of linear analysis to a tsunami lies in the range $0{\cdot}04 < k_c < 0{\cdot}10$. This range of wavenumbers corresponds to ocean wave periods of 18–50 min. A similar analysis for the initial characteristics given by (59*a*) yields a range of 13–24 min. It appears that many harbours have natural periods of oscillation which lie in or exceed this range (see, for example, Raichlen 1970). Hence linear analysis of a tsunami may be misleading in determining the potential excitation of resonance for these harbours.

5. Conclusions

The applicability of the following model equations for describing the evolution of long-wave initial data has been investigated: a linear non-dispersive model, a nonlinear non-dispersive model, a linear dispersive model and a nonlinear dispersive model. From this analysis the following conclusions can be stated.

(i) For the leading wave evolving from the initial data, the time of applicability for each model equation is governed by the magnitude of two non-dimensional parameters: (*a*) an Ursell number based on the amplitude and length of the initial data and (*b*) the volume of the initial data. The details are summarized in § 2.4.

(ii) For describing the trailing wave oscillations, which are inherently dispersive, a linear dispersive theory is applicable whenever a local Ursell number, based on the amplitude of the initial data and the local wavenumber, is small. When this parameter is $O(1)$ or larger, a nonlinear dispersive theory (e.g. the KdV equation) is required.

As a specific application of these modelling criteria the problem of tsunami propagation was examined. On the basis of assumed characteristics for a tsunami near its source region, the following conclusions appear to be applicable.

(iii) Asymptotic linear dispersive theory never applies to the propagation of tsunamis. If the length of the initial disturbance is approximately 100 miles, the leading wave ($k = 0$) is modelled by linear non-dispersive theory until the wave begins to shoal. For shorter lengths (~ 40 miles) linear dispersive theory is applicable.

(iv) The critical wavenumber k_c representing the lower limit of applicability of linear analysis to the oscillatory trailing waves lies in the range of $0{\cdot}4 < k_c < 0{\cdot}10$, which corresponds to ocean wave periods of 18–50 min for initial tsunamis characterized by (57*a*). A similar analysis using the characteristics (59*a*) yields a cut-off wave period in the range 13–34 min. For harbours whose fundamental period exceeds these ranges, the problem of harbour resonance requires a nonlinear dispersive model for the impinging tsunamis.

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